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Coexistence states for a diffusive one-prey and two-predators model with B–D functional response ☆

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A B S T R A C T

In this paper, we study a diffusive one-prey and two-predators system with Beddington–DeAngelis functional response. The sufficient and necessary conditions for the existence of coexistence states are obtained by means of the fixed point index theory. In addition, the stability and uniqueness of coexistence states are investigated. Finally, we give the sufficient conditions for extinction and permanence of the time-dependent system.

1. Introduction

The study of the dynamic relationship between predator and prey has long been one of the most important themes in population dynamics because of its universal existence in nature. One aspect of great interest for a predator–prey system is whether the various species can coexist. In many cases, the different species coexist in a steady state. When the species are homogeneously distributed, the existence and stability of the positive constant solutions to the mathematical model are important to study the dynamics of the system. In the spatially inhomogeneous case, the existence of a non-constant time-independent positive solution, also called stationary pattern, always indicates the dynamical richness of the system. In recent years, the existence of stationary pattern in various population dynamics models in the presence of diffusion has been studied extensively, and many different phenomena have been observed (see [3–15,23–27,41] and references therein).

In this paper, we are interested in a predator–prey system with one resource and two consumers. We assume that the two consumer species compete for the common resource following the Beddington–DeAngelis functional response. The model is a system of three differential equations of the form

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$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = u(r-u) - \dfrac{a_1 uv}{1+u+e_1 v} - \dfrac{a_2 uw}{1+u+e_2 w}, \\ \dfrac{\partial v}{\partial t} - \Delta v = \dfrac{m_1 uv}{1+u+e_1 v} - c_1 v, \\ \dfrac{\partial w}{\partial t} - \Delta w = \dfrac{m_2 uw}{1+u+e_2 w} - c_2 w & \text{in } \Omega \times (0,\infty), \\ k_1 \dfrac{\partial u}{\partial \nu} + u = k_2 \dfrac{\partial v}{\partial \nu} + v = k_3 \dfrac{\partial w}{\partial \nu} + w = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \big(u(0,x), v(0,x), w(0,x)\big) = \big(u_0(x), v_0(x), w_0(x)\big) \geqslant (0,0,0) & \text{in } \Omega \times (0,\infty), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N ($N \geqslant 1$ is an integer) with a smooth boundary $\partial\Omega$ and ν is the outward unit rector on $\partial\Omega$. The three functions u, v and w represent the densities of the prey and the two predators respectively. The positive constants m_1, m_2 represent the conversion rates of the prey to predators. The constants c_1 and c_2 are the death rates of the two predators. In this paper, we consider the more general case, which indicates that the constants c_1 and c_2 may change sign. The parameters r, a_i, e_i $(i = 1, 2)$ are strictly positive, and $k_i \geqslant 0$ $(i = 1, 2, 3)$.

The system (1.1) arising in mathematical biology as a predator–prey model describes three species interact each other and migrate in the same habitat Ω. The corresponding ODE system of (1.1) was proposed and studied in [26] when $e_1 = 0$, where the explanations of the ecological background can be found as well. In particular, the corresponding functional response $\frac{au}{1+u+ev}$ is called Beddington–DeAngelis functional response which was introduced by Beddington [1] and DeAngelis [2]. Compared to the well-known Holling–Tanner and Holling type-II functional response, it has an extra term in the denominator which models mutual interference between predators.

In the case that the predators $v = 0$ or $w = 0$, system (1.1) reduces to a two-species predator–prey model which has been studied extensively in the past several decades (see [3–15] and references therein). In particular, under Neumann and Robin boundary conditions, the authors in [3] mainly investigated the permanence and the existence of non-constant positive steady states in (1.1) while $w = 0$. For the same system as in [3], the authors in [8] established the existence of non-constant coexistence states under Robin boundary conditions by using topological degree theory. In [9] and [13], the authors discussed the uniqueness and exact multiplicity of the positive steady-state solutions in the similar system of (1.1) with the Dirichlet boundary conditions while $w = 0$.

On the other hand, there are not many works on the dynamics of three-species reaction–diffusion systems since they are much more complicated than those of two-species cases. These include three-species or n-species Lotka–Volterra competition model [19–22], one-predator two-prey model [23–25], one-prey two-predators model [27], food-chains model [28–31]. For more dynamics of three or more species interacting systems, one can refer to [32–37] and references therein. We remark that problem (1.1) with homogeneous Neumann boundary conditions was discussed in [27] recently when $e_1 = 0$. We also point out that, to our knowledge, little work has been done about problem (1.1).

In our work here, one of the main purposes is to study the existence of positive stationary solutions of (1.1) by using fixed point index theory, which are the solutions of

$$\begin{cases} -\Delta u = u(r-u) - \dfrac{a_1 uv}{1+u+e_1 v} - \dfrac{a_2 uw}{1+u+e_2 w}, \\ -\Delta v = \dfrac{m_1 uv}{1+u+e_1 v} - c_1 v, \\ -\Delta w = \dfrac{m_2 uw}{1+u+e_2 w} - c_2 w & \text{in } \Omega, \\ k_1 \dfrac{\partial u}{\partial \nu} + u = k_2 \dfrac{\partial v}{\partial \nu} + v = k_3 \dfrac{\partial w}{\partial \nu} + w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Hence we are interested in positive solutions of (1.2), which correspond to coexistence states of prey and predators.

The rest of this paper is organized as follows. In Section 2, some necessary preliminaries are introduced. In Section 3, we give a priori upper bounds for positive solutions and investigate the existence and nonexistence of coexistence states of model (1.2) by using some degree theorems developed. In Section 4, the stability and uniqueness of coexistence states of (1.2) are studied. Finally, in Section 5, some sufficient conditions for the extinction and permanence of the time-dependent system (1.1) are obtained.

2. Some preliminaries

In order to give our results and complete the corresponding proofs, we introduce some necessary notations and fundamental theorems in this section, which play an important role in rest of this paper.

For each $h(x) \in C^{\alpha}(\Omega)$ $(0 < \alpha < 1)$ and $k \geqslant 0$, let $\lambda_{1,k}(h(x))$ denote the principle eigenvalue of the following eigenvalue problem

$$\begin{cases} -\Delta u + h(x)u = \lambda u & \text{in } \Omega, \\ k\dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

and denote $\lambda_{1,k}(0)$ by $\lambda_{1,k}$ for simplicity. It is known that $\lambda_{1,k}(h(x))$ is strictly increasing in the sense that $h_1(x) \leqslant h_2(x)$ and $h_1(x) \not\equiv h_2(x)$ implies that $\lambda_{1,k}(h_1(x)) < \lambda_{1,k}(h_2(x))$.

In order to calculate the indexes at the trivial and semi-trivial states by using the fixed point index theory, we introduce the following theorem.

Theorem 2.1. *(See [15,16,49].) Assume $h(x) \in C^\alpha(\Omega)$ $(0 < \alpha < 1)$ and M is a sufficiently large number such that $M > h(x)$ for all $x \in \overline{\Omega}$. Define a positive and compact operator $\mathbb{L} := (-\Delta + M)^{-1}(M - h(x)) : C_k^1(\overline{\Omega}) \to C_k^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}):\ k\frac{\partial u}{\partial \nu} + u = 0 \text{ on } \partial\Omega\}$ for $k \geqslant 0$. Denote the spectral radius of $\mathbb{L}$ by $r_k(\mathbb{L})$.*

(i) $\lambda_{1,k}(h) > 0$ *if and only if* $r(\mathbb{L}) < 1$.
(ii) $\lambda_{1,k}(h) < 0$ *if and only if* $r(\mathbb{L}) > 1$.
(iii) $\lambda_{1,k}(h) = 0$ *if and only if* $r(\mathbb{L}) = 1$.

From Theorem 2.1, we see that it is crucial to know the sign of the eigenvalue $\lambda_{1,k}(h)$ to determine the spectral radius of $\mathbb{L}$. The following theorem is established by Theorem 2.4 of [38] (see also [39,40,45]), which gives some sufficient conditions to determine the sign of the eigenvalue $\lambda_{1,k}(h)$.

Theorem 2.2. *(See [38–40,45].) Let $h(x) \in L^\infty(\Omega)$ and $\varphi \geqslant 0$, $\varphi \not\equiv 0$ in Ω with $k\frac{\partial \varphi}{\partial \nu} + \varphi = 0$ on $\partial\Omega$ for $k \geqslant 0$.*

(i) *If* $0 \not\equiv -\Delta\varphi + h(x)\varphi \leqslant 0$, *then* $\lambda_1(h(x)) < 0$.
(ii) *If* $0 \not\equiv -\Delta\varphi + h(x)\varphi \geqslant 0$, *then* $\lambda_1(h(x)) > 0$.
(iii) *If* $-\Delta\varphi + h(x)\varphi \equiv 0$, *then* $\lambda_1(h(x)) = 0$.

Consider the following equation

$$\begin{cases} -\Delta u = u f(x,u) & \text{in } \Omega, \\ k\dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where Ω is a bounded domain in R^N ($N \geqslant 1$ is an integer) with a smooth boundary $\partial\Omega$, k is nonnegative constant, ν is the outward unit rector on $\partial\Omega$. Assume that the function $f(x,u) : \overline{\Omega} \times [0,\infty) \to R$ satisfies the following hypotheses:

(H1) $f(x,u)$ is C^α-function in x, where $0 < \alpha < 1$;
(H2) $f(x,u)$ is C^1-function in u with $f_u(x,u) < 0$ for all $(x,u) \in \overline{\Omega} \times [0,\infty)$;
(H3) $f(x,u) \leqslant 0$ on $(x,u) \in \overline{\Omega} \times [C,\infty)$ for some positive constant C.

Theorem 2.3. *(See [45,47].)*

(i) *The nonnegative solution $u(x)$ of* (2.2) *satisfies $u(x) \leqslant C$ for all $x \in \overline{\Omega}$.*
(ii) *If $\lambda_{1,k}(-f(x,0)) \geqslant 0$, then* (2.2) *has no positive solutions. Moreover, the trivial solution $u(x) = 0$ is globally asymptotically stable.*
(iii) *If $\lambda_{1,k}(-f(x,0)) < 0$, then* (2.2) *has a unique positive solution which is globally asymptotically stable. In this case, the trivial solution $u(x) = 0$ is unstable.*

Let $\Theta_k(\rho(x))$ with $\Theta_k(\rho(x)) \leqslant \max_{x\in\overline{\Omega}}(\rho(x))$ be the unique positive solution of the following equation

$$\begin{cases} -\Delta\varphi = \varphi\bigl(\rho(x) - \varphi\bigr) & \text{in } \Omega, \\ k\dfrac{\partial \varphi}{\partial \nu} + \varphi = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

if $\lambda_{1,k}(\rho(x)) < 0$, where $\rho(x) \in C^\alpha(\Omega)$ $(0 < \alpha < 1)$ is a positive function. We can see that the existence of the positive solution of (2.3) follows from Theorem 2.3.

Now, we introduce the fixed point index theory which plays an important role in getting the sufficient conditions for the existence of coexistence states of model (1.2).

Let $\mathbb{E}$ be a real Banach space and $\mathbb{W} \subset \mathbb{E}$ be the natural positive cone of $\mathbb{E}$. For $y \in \mathbb{W}$, define $\mathbb{W}_y = \{x \in \mathbb{E}:\ y + \gamma \in \mathbb{W}$ for some $\gamma > 0\}$ and $S_y = \{x \in \overline{\mathbb{W}_y}:\ -x \in \overline{\mathbb{W}_y}\}$. Then $\overline{\mathbb{W}_y}$ is a wedge containing $\mathbb{W}$, y, $-y$, while S_y is a closed subset of $\mathbb{E}$ containing y. Let T be a compact linear operator on $\mathbb{E}$ which satisfies $T(\overline{\mathbb{W}_y}) \subset \overline{\mathbb{W}_y}$. We say that T has property α on $\overline{\mathbb{W}_y}$ if there is a $t \in (0,1)$ and an $\omega \in \overline{\mathbb{W}_y} \setminus S_y$ such that $(I - tT)\omega \in S_y$. Assume $\mathcal{A} : \mathbb{W} \to \mathbb{W}$ is a compact operator with a fixed point $y \in \mathbb{W}$ and $\mathcal{A}$ is a Fréchet differentiable at y. Let $\mathbb{L} = \mathcal{A}'(y)$ be the Fréchet derivative of $\mathcal{A}$ at y. Then $\mathbb{L}$ maps $\overline{\mathbb{W}_y}$ into itself. We denote by $\deg_{\mathbb{W}}(I - \mathcal{A}, \mathbb{D})$ the degree of $I - \mathcal{A}$ in $\mathbb{D}$ relative to $\mathbb{W}$, by $\mathrm{index}_{\mathbb{W}}(\mathcal{A}, y)$ the fixed point index of $\mathcal{A}$ at y relative to $\mathbb{W}$ and $\deg_{\mathbb{W}}(I - \mathcal{A}, S) = \sum_{y \in S} \mathrm{index}_{\mathbb{W}}(\mathcal{A}, y)$ where S only contains discrete points. Then the following theorem can be obtained.

Theorem 2.4. *(See [12,18,25].) Assume that $I - \mathbb{L}$ is invertible on $\mathbb{W}_y$.*

(i) *If $\mathbb{L}$ has property α on $\mathbb{W}_y$, then* $\mathrm{index}_{\mathbb{W}}(\mathcal{A}, y) = 0$.
(ii) *If $\mathbb{L}$ does not have property α on $\mathbb{W}_y$, then* $\mathrm{index}_{\mathbb{W}}(\mathcal{A}, y) = (-1)^{\sigma}$, *where σ is the sum of algebraic multiplicities of the eigenvalues of $\mathbb{L}$ which are greater than* 1.

Finally, we introduce the following theorem about degree calculations, which was introduced by E.N. Dancer and Y.H. Du in [19] and we state here for convenience.

Assume that $\mathbb{E}_1$ and $\mathbb{E}_2$ are ordered Banach spaces with positive cones $\mathbb{W}_1$ and $\mathbb{W}_2$, respectively. Let $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$ and $\mathbb{W} = \mathbb{W}_1 \oplus \mathbb{W}_2$. Then $\mathbb{E}$ is an ordered Banach space with positive cone $\mathbb{W}$. Let $\mathbb{D}$ be an open set in $\mathbb{W}$ containing 0 and $\Re_i := \overline{\mathbb{D}} \to \mathbb{W}_i$ be compact operators, $i = 1, 2$. Assume (u, v) is a general element in $\mathbb{W}$ with $u \in \mathbb{W}_1$ and $v \in \mathbb{W}_2$. Define $\Re := \overline{\mathbb{D}} \to \mathbb{W}$ by $\Re(u, v) = (\Re_1(u, v), \Re_2(u, v))$ and $\mathbb{W}_2(\delta) = \{v \in \mathbb{W}_2\colon \|v\|_{\mathbb{E}_2} < \delta\}$.

Then we have the following theorem:

Theorem 2.5. *Suppose $U \subset \mathbb{W}_1 \cap \mathbb{D}$ is relatively open and bounded, and $\Re_1(u, 0) \neq u$ for $u \in \partial U$, $\Re_2(u, 0) \equiv 0$ for $u \in \overline{U}$. Suppose $\Re_2 := \overline{\mathbb{D}} \to \mathbb{W}_2$ extends to a continuously differentiable mapping of a neighborhood of $\mathbb{D}$ into $\mathbb{E}_2$, $\mathbb{W}_2 - \mathbb{W}_2$ is dense in $\mathbb{E}_2$ and $S = \{u \in U\colon u = \Re_1(u, 0)\}$.*

(i) *If for any $u \in S$, the spectral radius $r(\Re_2'(u, 0))|_{\mathbb{W}_2} > 1$ and 1 is not an eigenvalue of $\Re_2'(u, 0)|_{\mathbb{W}_2}$ corresponding to a positive eigenvector, then* $\deg_{\mathbb{W}}(I - \Re, U \times \mathbb{W}_2(\delta), 0) = 0$ *for $\delta > 0$ small.*
(ii) *If for any $u \in S$, the spectral radius $r(\Re_2'(u, 0))|_{\mathbb{W}_2} < 1$, then* $\deg_{\mathbb{W}}(I - \Re, U \times \mathbb{W}_2(\delta), 0) = \deg_{\mathbb{W}}(I - \Re_1|_{\mathbb{W}_1}, U, 0)$ *for $\delta > 0$ small.*

3. Existence of coexistence states

In this section, we start with the following lemma which gives the necessary conditions for the existence of coexistence states for system (1.2).

Theorem 3.1. *If problem* (1.2) *has a coexistence state, then $r > \lambda_{1,k_1}$, $\lambda_{1,k_2}(-\frac{m_1 \Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}) < -c_1 < \lambda_{1,k_2}$ and $\lambda_{1,k_3}(-\frac{m_2 \Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}) < -c_2 < \lambda_{1,k_3}$.*

Proof. Assume (u, v, w) is a coexistence state of (1.2). Then it is easy to see that $r = \lambda_{1,k_1}(u + \frac{a_1 v}{1+u+e_1 v} + \frac{a_2 w}{1+u+e_2 w}) > \lambda_{1,k_1}$ by the comparison principle of eigenvalues. Since

$$\begin{cases} -\Delta u = u(r-u) - \dfrac{a_1 u v}{1+u+e_1 v} - \dfrac{a_2 u w}{1+u+e_2 w} < u(r-u) & \text{in } \Omega, \\ k_1 \dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

we can obtain $u(x) < \Theta_{k_1}(r)$ by comparison principle. From the second equation of (1.2), by using the comparison principle of eigenvalues again, we can get

$$0 = \lambda_{1,k_2}\left(c_1 - \frac{m_1 u}{1+u+e_1 v}\right) < \lambda_{1,k_2}(c_1) = c_1 + \lambda_{1,k_2}$$

and

$$0 = \lambda_{1,k_2}\left(c_1 - \frac{m_1 u}{1+u+e_1 v}\right) > \lambda_{1,k_2}\left(c_1 - \frac{m_1 \Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\right) = c_1 + \lambda_{1,k_2}\left(-\frac{m_1 \Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\right).$$

Thus, $\lambda_{1,k_2}(-\frac{m_1 \Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}) < -c_1 < \lambda_{1,k_2}$. Similarly, we can prove that $\lambda_{1,k_3}(-\frac{m_2 \Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}) < -c_2 < \lambda_{1,k_3}$. The proof is completed. □

Remark 3.2. From the lemma above, we can see that (1.2) has no coexistence states if one of the following conditions holds:

(i) $r \leqslant \lambda_{1,k_1}$;
(ii) $r > \lambda_{1,k_1}$ and $-c_1 \leqslant \lambda_{1,k_2}(-\frac{m_1 \Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$;

(iii) $r > \lambda_{1,k_1}$ and $-c_2 \leqslant \lambda_{1,k_3}(-\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$;
(iv) $-c_1 \geqslant \lambda_{1,k_2}$ or $-c_2 \geqslant \lambda_{1,k_3}$.

In the rest of this section, we shall give sufficient conditions for (1.2) to have coexistence states by using the fixed point index theory. So, it is necessary to get the a priori bounds for the coexistence states of (1.2).

Lemma 3.3. *If $m_1 r > c_1(1+r)$ and $m_2 r > c_2(1+r)$, then any coexistence state (u, v, w) of* (1.2) *has an a priori bounds*:

$$u(x) \leqslant Q_1, \qquad v(x) \leqslant Q_2, \qquad w(x) \leqslant Q_3,$$

where

$$Q_1 = r, \qquad Q_2 = \frac{m_1 r - c_1(1+r)}{c_1 e_1}, \qquad Q_3 = \frac{m_2 r - c_2(1+r)}{c_2 e_2}.$$

Proof. From the first equation of (1.2), we can obtain $u(x) \leqslant r$ easily by the maximum principle. Then from the second equation of (1.2), we can get that

$$\begin{cases} -\Delta v \leqslant v\left(\dfrac{m_1 r}{1+r+e_1 v} - c_1\right) & \text{in } \Omega, \\ k\dfrac{\partial v}{\partial \nu} + v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Hence, by using the maximum principle again, we have

$$v(x) \leqslant \frac{m_1 r - c_1(1+r)}{c_1 e_1}.$$

Similarly, we can prove that

$$w(x) \leqslant \frac{m_2 r - c_2(1+r)}{c_2 e_2}.$$

The proof is completed. □

Remark 3.4. From the proof of Lemma 3.3, we can see that $m_1 r > c_1(1+r)$ and $m_2 r > c_2(1+r)$ are the necessary conditions for problem (1.2) to have coexistence states. So, throughout this subsection, we assume that: (H) $m_1 r > c_1(1+r)$, and $m_2 r > c_2(1+r)$.

Now, we introduce the following notations:

$$\begin{aligned} &\mathbb{E} = C^1_{k_1}(\overline{\Omega}) \oplus C^1_{k_2}(\overline{\Omega}) \oplus C^1_{k_3}(\overline{\Omega}), \\ &\mathbb{N}_i = \left\{\varphi \in C^1_{k_i}(\overline{\Omega}):\ \varphi \geqslant 0 \text{ in } \overline{\Omega}\right\}, \quad i = 1, 2, 3, \\ &\mathbb{W} = \mathbb{N}_1 \oplus \mathbb{N}_2 \oplus \mathbb{N}_3, \\ &\mathbb{D} = \left\{(u, v, w) \in \mathbb{W}:\ u \leqslant (Q_1+1),\ v \leqslant (Q_2+1),\ w \leqslant (Q_3+1)\right\}, \end{aligned}$$

where $C^1_{k_i}(\overline{\Omega}) = \{\phi \in C^1(\overline{\Omega}):\ k_i\frac{\partial\phi}{\partial\nu} + \phi = 0 \text{ on } \partial\Omega,\ i = 1, 2, 3\}$ and Q_1, Q_2, Q_3 are defined in Lemma 3.3.

From Lemma 3.3, we can see that the coexistence state of (1.2) must be in $\mathbb{D}$. Take q sufficiently large with $q > \max\{r + \frac{a_1}{e_1}Q_2 + \frac{a_2}{e_2}Q_3, c_1, c_2\}$ such that $u(r-u) - \frac{a_1 uv}{1+u+e_1 v} - \frac{a_2 uw}{1+u+e_2 w} + qu$, $\frac{m_1 uv}{1+u+e_1 v} - c_1 v + qv$ and $\frac{m_2 uw}{1+u+e_1 w} - c_2 w + qw$ are respectively monotone increasing with respect to u, v and w for all $(u, v, w) \in [0, Q_1] \times [0, Q_2] \times [0, Q_3]$.

Define a positive and compact operator $\Re : \mathbb{E} \to \mathbb{E}$ by

$$\Re(u, v, w) = (-\Delta + q)^{-1}\begin{pmatrix} u(r - u - \frac{a_1 uv}{1+u+e_1 v} - \frac{a_2 uw}{1+u+e_2 w}) + qu \\ v(\frac{m_1 u}{1+u+e_1 v} - c_1) + qv \\ w(\frac{m_2 u}{1+u+e_2 w} - c_2) + qw \end{pmatrix}.$$

Remark 3.5. Observe that (1.2) is equivalent to $(u, v, w) = \Re(u, v, w)$, and then it is sufficient to prove that $\Re$ has a non-constant positive fixed point in $\mathbb{D}$ to show that (1.2) has a coexistence state.

From the remark above, we can see that it is necessary to calculate the degree of $I - \Re$ in $\mathbb{D}$ relative to $\mathbb{W}$ and the fixed point index of $\Re$ at $(0, 0, 0)$ relative to $\mathbb{W}$. The following lemma gives the corresponding results about $\deg_{\mathbb{W}}(I - \Re, \mathbb{D})$ and $\text{index}_W(\Re, (0, 0, 0))$.

Lemma 3.6.

(i) $\deg_{\mathbb{W}}(I-\Re,\mathbb{D})=1$.
(ii) *If* $r>\lambda_{1k_1}$, $c_1>-\lambda_{1,k_2}$ *and* $c_2>-\lambda_{1,k_3}$, *then* $\operatorname{index}_W(\Re,(0,0,0))=0$.

Proof. (i) It is easy to see that $\Re$ has no fixed point on $\partial\mathbb{D}$. So, $\deg_{\mathbb{W}}(I-\Re,\mathbb{D})$ is well defined. For $\mu\in[0,1]$, define a positive and compact operator $\Re_\mu:\mathbb{E}\to\mathbb{E}$ by

$$\Re_\mu(u,v,w)=(-\Delta+q)^{-1}\begin{pmatrix}\mu u(r-u-\frac{a_1v}{1+u+e_1v}-\frac{a_2w}{1+u+e_2w})+qu\\ \mu v(\frac{m_1u}{1+u+e_1v}-c_1)+qv\\ \mu w(\frac{m_2u}{1+u+e_1w}-c_2)+qw\end{pmatrix}.$$

Then $\Re_1=\Re$ and a fixed point of $\Re_\mu$ is a solution of the following problem

$$\begin{cases}-\Delta u=\mu u\left(r-u-\dfrac{a_1v}{1+u+e_1v}-\dfrac{a_2w}{1+u+e_2w}\right),\\ -\Delta v=\mu\left(\dfrac{m_1uv}{1+u+e_1v}-c_1v\right),\\ -\Delta w=\mu\left(\dfrac{m_2uw}{1+u+e_2w}-c_2w\right) & \text{in } \Omega,\\ k_1\dfrac{\partial u}{\partial\nu}+u=k_2\dfrac{\partial v}{\partial\nu}+v=k_3\dfrac{\partial w}{\partial\nu}+w=0 & \text{on } \partial\Omega.\end{cases}\tag{3.3}$$

As in the proof of Lemma 3.3, we can show that the fixed point (u,v,w) of $\Re_\mu$ also satisfies $u\leqslant Q_1$, $v\leqslant Q_2$ and $w\leqslant Q_3$ for each $\mu\in[0,1]$. Thus, $\Re_\mu$ has no fixed point on $\partial\mathbb{D}$ and $\deg_{\mathbb{W}}(I-\Re_\mu,\mathbb{D})$ is well defined. Since $\deg_{\mathbb{W}}(I-\Re_\mu,\mathbb{D})$ is independent of μ, we have $\deg_{\mathbb{W}}(I-\Re,\mathbb{D})=\deg_{\mathbb{W}}(I-\Re_1,\mathbb{D})=\deg_{\mathbb{W}}(I-\Re_0,\mathbb{D})$.

Observe that (3.3) has only the trivial solution $(0,0,0)$ when $\mu=0$. Set

$$\mathbb{L}=\Re_0'(0,0,0)=(-\Delta+q)^{-1}\begin{pmatrix}q&0&0\\0&q&0\\0&0&q\end{pmatrix}.$$

Assume that $\mathbb{L}(\xi_1,\xi_2,\xi_3)=(\xi_1,\xi_2,\xi_3)$ for some $(\xi_1,\xi_2,\xi_3)\in\overline{\mathbb{W}}_{(0,0,0)}=\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. It is easy to show that $(\xi_1,\xi_2,\xi_3)=(0,0,0)$ by maximum principle. So, $I-\mathbb{L}$ is invertible on $\overline{\mathbb{W}}_{(0,0,0)}$. Since $\lambda_{1,k_i}>0$, we have $r_{k_i}(\mathbb{L})<1$ for $i=1,2,3$ by Theorem 2.1. This implies that $\mathbb{L}$ does not have property α. So, by Theorem 2.4, we have $\deg_{\mathbb{W}}(I-\Re,\mathbb{D})=\deg_{\mathbb{W}}(I-\Re_0,\mathbb{D})=\operatorname{index}_{\mathbb{W}}(I-\Re_0,(0,0,0))=1$.

(ii) Note that $\Re(0,0,0)=(0,0,0)$. Let $\mathbb{L}=\Re'(0,0,0)$ and then

$$\mathbb{L}=(-\Delta+q)^{-1}\begin{pmatrix}r+q&0&0\\0&-c_1+q&0\\0&0&-c_2+q\end{pmatrix}.$$

Assume that $\mathbb{L}(\xi_1,\xi_2,\xi_3)=(\xi_1,\xi_2,\xi_3)$ for some $(\xi_1,\xi_2,\xi_3)\in\overline{\mathbb{W}}_{(0,0,0)}=\mathbb{N}\times\mathbb{N}\times\mathbb{N}$. Then

$$\begin{cases}-\Delta\xi_1=r\xi_1,\\ -\Delta\xi_2=-c_1\xi_2,\\ -\Delta\xi_3=-c_2\xi_3 & \text{in } \Omega,\\ k_1\dfrac{\partial\xi_1}{\partial\nu}+\xi_1=k_2\dfrac{\partial\xi_2}{\partial\nu}+\xi_2=k_3\dfrac{\partial\xi_3}{\partial\nu}+\xi_3=0 & \text{on } \partial\Omega.\end{cases}\tag{3.4}$$

Since $r>\lambda_{1,k_1}$, we can show $\xi_1\equiv0$. If not, we have $\lambda_{1,k_1}=r$ from the first equation of (3.4), which is a contradiction. Similarly, since $c_1>-\lambda_{1,k_2}$ and $c_2>-\lambda_{1,k_3}$, we can get $\xi_2\equiv0$ and $\xi_3\equiv0$. So, $(\xi_1,\xi_2,\xi_3)\equiv(0,0,0)$ and $I-\mathbb{L}$ is invertible on $\overline{\mathbb{W}}_{(0,0,0)}$.

Note that $r>\lambda_{1,k_1}$ implies $r_0=r_{k_1}[(-\Delta+q)^{-1}(r+q)]>1$ by Theorem 2.1, where r_0 is the principle eigenvalue of the operator $(-\Delta+q)^{-1}(r+q)$ with a corresponding eigenfunction $\phi(x)>0$; we have $(\phi(x),0,0)\in\overline{\mathbb{W}}_{(0,0,0)}\setminus S_{(0,0,0)}$ since $S_{(0,0,0)}=(0,0,0)$. Then $(I-r_0^{-1}\mathbb{L})(\phi(x),0,0)=(0,0,0)\in S_{(0,0,0)}$, which shows that $\mathbb{L}$ has property α. Therefore, $\operatorname{index}_{\mathbb{W}}(\Re,(0,0,0))=0$ by Theorem 2.4. The proof is completed. $\square$

The next lemma gives the index at the semi-trivial solution $(\Theta_{k_1}(r),0,0)$ of (1.2).

Lemma 3.7. *Assume that* $r > \lambda_{1k_1}$, $-c_1 \neq \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$ *and* $-c_2 \neq \lambda_{1,k_3}(-\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$.

(i) $\mathrm{index}_{\mathbb{W}}(\Re, (\Theta_{k_1}(r), 0, 0)) = 0$ *if* $-c_1 > \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$ *or* $-c_2 > -\lambda_{1,k_3}(-\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$;

(ii) $\mathrm{index}_{\mathbb{W}}(\Re, (\Theta_{k_1}(r), 0, 0)) = 1$ *if* $-c_1 < \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$ *and* $-c_2 < \lambda_{1,k_3}(-\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$.

Proof. (i) Note that $\Re(\Theta_{k_1}(r), 0, 0) = (\Theta_{k_1}(r), 0, 0)$. Let $\mathbb{L} = \Re'(\Theta_{k_1}(r), 0, 0)$ and then

$$\mathbb{L} = (-\Delta + q)^{-1}\begin{pmatrix} r - 2\Theta_{k_1}(r) + q & -\frac{a_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)} & -\frac{a_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)} \\ 0 & -\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)} - c_1 + q & 0 \\ 0 & 0 & -\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)} - c_2 + q \end{pmatrix}.$$

If $\mathbb{L}(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_2, \xi_3)$ for some $(\xi_1, \xi_2, \xi_3) \in \overline{\mathbb{W}}_{(\Theta_{k_1}(r),0,0)} = C^1_{k_1}(\overline{\Omega}) \times \mathbb{N} \times \mathbb{N}$, then

$$\begin{cases} -\Delta\xi_1 + \big(2\Theta_{k_1}(r) - r\big)\xi_1 = -\dfrac{a_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\xi_2 - \dfrac{a_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\xi_3, & \\ -\Delta\xi_2 + \left(c_1 - \dfrac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\right)\xi_2 = 0, & \\ -\Delta\xi_3 + \left(c_2 - \dfrac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\right)\xi_3 = 0 & \text{in } \Omega, \\ k_1\dfrac{\partial\xi_1}{\partial\nu} + \xi_1 = k_2\dfrac{\partial\xi_2}{\partial\nu} + \xi_2 = k_3\dfrac{\partial\xi_3}{\partial\nu} + \xi_3 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Taking account of $\xi_2 \in \mathbb{N}$, if $\xi_2 \neq 0$, we can see from the second equation of (3.5) that $-c_1 = \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$. This contradicts $-c_1 \neq \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$. So, $\xi_2 \equiv 0$. Similarly, we can prove that $\xi_3 \equiv 0$. Then from the first equation of (3.5), we can get that

$$\begin{cases} -\Delta\xi_1 + \big(2\Theta_{k_1}(r) - r\big)\xi_1 = 0 & \text{in } \Omega, \\ k_1\dfrac{\partial\xi_1}{\partial\nu} + \xi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

If $\xi_1 \neq 0$, then $\lambda_{1k_1}(2\Theta_{k_1}(r) - r) = 0$. On the other hand, $\lambda_{1k_1}(2\Theta_{k_1}(r) - r) > \lambda_{1k_1}(\Theta_{k_1}(r) - r) = 0$, which is a contradiction. Therefore, $(\xi_1, \xi_2, \xi_3) \equiv (0, 0, 0)$ and $I - \mathbb{L}$ is invertible on $\overline{\mathbb{W}}_{(\Theta,0,0)}$.

We claim that $\mathbb{L}$ has property α on $\overline{\mathbb{W}}_{(\Theta_{k_1}(r),0,0)}$. In fact, set

$$A = (-\Delta + q)^{-1}\left(\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)} - c_1 + q\right).$$

Since $-c_1 > \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$, we can see that $r_{c_1}(A) > 1$ is an eigenvalue of A with a corresponding eigenfunction $\phi_{c_1}(x) > 0$ by Theorem 2.1. Noting that $S_{(\Theta_{k_1}(r),0,0)} = C^1_{k_1}(\overline{\Omega}) \times \{0\} \times \{0\}$, we know $(0, \phi_{c_1}, 0) \in \overline{\mathbb{W}}_{(\Theta_{k_1}(r),0,0)} \setminus S_{(\Theta_{k_1}(r),0,0)}$. Then we have

$$\begin{aligned} \big(I - r_{c_1}^{-1}\mathbb{L}\big)\begin{pmatrix} 0 \\ \phi_{c_1} \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ \phi_{c_1} \\ 0 \end{pmatrix} - r_{c_1}^{-1}(-\Delta + q)^{-1}\begin{pmatrix} -\frac{a_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\phi_{c_1} \\ (\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}}(r) - c_1 + q)\phi_{c_1} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} r_{c_1}^{-1}(-\Delta + q)^{-1}\frac{a_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\phi_{c_1} \\ 0 \\ 0 \end{pmatrix} \in S_{(\Theta,0,0)}. \end{aligned} \tag{3.7}$$

This establishes our claim. Hence, $\mathrm{index}_{\mathbb{W}}(\Re, (\Theta_{k_1}(r), 0, 0)) = 0$ by Theorem 2.4.

(ii) First, we show that $\mathbb{L}$ has no property α in $\overline{\mathbb{W}}_{(\Theta_{k_1}(r),0,0)}$. On the contrary, if $\mathbb{L}$ has property α in $\overline{\mathbb{W}}_{(\Theta_{k_1}(r),0,0)}$, then there exist $\gamma \in (0, 1)$ and $(\varphi_1, \varphi_2, \varphi_3) \in \overline{\mathbb{W}}_{(\Theta_{k_1}(r),0,0)} \setminus S_{(\Theta_{k_1}(r),0,0)}$ such that $(I - \gamma\mathbb{L})(\varphi_1, \varphi_2, \varphi_3) \in S_{(\Theta_{k_1}(r),0,0)}$. Therefore, $(-\Delta + q)^{-1}(\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)} - c_1 + q)\varphi_2 = \frac{1}{\gamma}\varphi_2$, which implies that $\frac{1}{\gamma} > 1$ is an eigenvalue of the operator $(-\Delta + q)^{-1}(\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)} - c_1 + q)$.

On the other hand, since $-c_1 < \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$, we have $r_{k_2}((-\Delta+q)^{-1}(\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}-c_1+q))<1$. This contradiction shows that $\mathbb{L}$ does not have property α on $\overline{\mathbb{W}}_{(\Theta_{k_1}(r),0,0)}$. So by Theorem 2.4, we have

$$\operatorname{index}_W\big(\Re,\big(\Theta_{k_1}(r),0,0\big)\big)=(-1)^{\sigma},$$

where σ is the sum of the multiplicities of all eigenvalues of $\mathbb{L}$ which are greater that 1.

Next, we shall prove that $\sigma=0$. Suppose $\mu>1$ is an eigenvalue of $\mathbb{L}$ with corresponding eigenfunction (ξ_1,ξ_2,ξ_3), then we have

$$\begin{cases}-\Delta\xi_1+q\xi_1=\dfrac{1}{\mu}\left(\big(r-2\Theta_{k_1}(r)+q\big)\xi_1-\dfrac{a_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\xi_2-\dfrac{a_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\xi_3\right),\\ -\Delta\xi_2+q\xi_2=\dfrac{1}{\mu}\left(\dfrac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}-c_1+q\right)\xi_2,\\ -\Delta\xi_3+q\xi_3=\dfrac{1}{\mu}\left(\dfrac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}-c_2+q\right)\xi_3 & \text{in } \Omega,\\ k_1\dfrac{\partial\xi_1}{\partial\nu}+\xi_1=k_2\dfrac{\partial\xi_2}{\partial\nu}+\xi_2=k_3\dfrac{\partial\xi_3}{\partial\nu}+\xi_3=0 & \text{on } \partial\Omega.\end{cases}\tag{3.8}$$

If $\xi_2\neq 0$, we can get from the second equation of (3.8) that

$$0=\lambda_{1,k_2}\left(q\left(1-\frac{1}{\mu}\right)-\frac{1}{\mu}\left(\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}-c_1\right)\right)>\lambda_{1,k_2}\left(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}+c_1\right)=\lambda_{1,k_2}\left(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\right)+c_1.$$

This contradicts $-c_1<\lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$. So, $\xi_2\equiv 0$. Similarly, we can prove that $\xi_3\equiv 0$. Then from the first equation of (3.8), we can get that

$$0=\lambda_{1,k_1}\left(q\left(1-\frac{1}{\mu}\right)-\frac{1}{\mu}\big(r-2\Theta_{k_1}(r)\big)\right)\geqslant\lambda_{1,k_1}\big(2\Theta_{k_1}(r)-r\big)>\lambda_{1,k_1}\big(\Theta_{k_1}(r)-r\big)=0.$$

This contradiction shows that $\xi_1\equiv 0$. So, $(\xi_1,\xi_2,\xi_3)\equiv(0,0,0)$, which implies that $\mathbb{L}$ has no eigenvalue being greater than 1. Consequently, $\sigma=0$ and then $\operatorname{index}_W(\Re,(\Theta_{k_1}(r),0,0))=1$ by Theorem 2.4. The proof is completed. □

To study the other semi-trivial solutions of (1.2), consider the following three possible subsystems:

$$\begin{cases}-\Delta u=u(r-u)-\dfrac{a_1uv}{1+u+e_1v},\\ -\Delta v=\dfrac{m_1uv}{1+u+e_1v}-c_1v & \text{in } \Omega,\\ k_1\dfrac{\partial u}{\partial\nu}+u=k_2\dfrac{\partial v}{\partial\nu}+v=0 & \text{on } \partial\Omega,\end{cases}\tag{3.9}$$

$$\begin{cases}-\Delta u=u(r-u)-\dfrac{a_2uw}{1+u+e_2w},\\ -\Delta w=\dfrac{m_2uw}{1+u+e_2w}-c_2w & \text{in } \Omega,\\ k_1\dfrac{\partial u}{\partial\nu}+u=k_3\dfrac{\partial w}{\partial\nu}+w=0 & \text{on } \partial\Omega,\end{cases}\tag{3.10}$$

and

$$\begin{cases}-\Delta v=-c_1v,\\ -\Delta w=-c_2w & \text{in } \Omega,\\ k_2\dfrac{\partial v}{\partial\nu}+v=k_3\dfrac{\partial w}{\partial\nu}+w=0 & \text{on } \partial\Omega.\end{cases}\tag{3.11}$$

It is easy to see that system (3.11) has only trivial solution $(v,w)=(0,0)$. About the positive solutions of systems (3.9) and (3.10), we can obtain some results from [41–44] and simple comparison arguments for elliptic problems. We point out that the corresponding main results are still valid under Robin boundary conditions even if the results in the above references were obtained under Dirichlet boundaries.

Theorem 3.8. *(3.9) has a positive solution (u^*, v^*) with $u^* \leqslant \Theta_{k_1}(r)$ if and only if $r > \lambda_{1,k_1}$ and $-c_1 > \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$. In addition, if $r - \frac{a_1}{e_1} > \lambda_{1,k_1}$ and $-c_1 > \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r-\frac{a_1}{e_1})}{1+\Theta_{k_1}(r-\frac{a_1}{e_1})})$, then the positive solution (u^*, v^*) satisfies $\Theta_{k_1}(r - \frac{a_1}{e_1}) \leqslant u^*$ and $\tilde{v} \leqslant v^*$, where $\tilde{v}$ is the unique positive solution of the following problem*

$$\begin{cases} -\Delta v = v\left(\dfrac{m_1\Theta_{k_1}(r-\frac{a_1}{e_1})}{1+\Theta_{k_1}(r-\frac{a_1}{e_1})+e_1 v} - c_1\right) & \text{in } \Omega, \\ k_2\dfrac{\partial v}{\partial \nu} + v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Theorem 3.9. *(3.10) has a positive solution $(u^\sharp, w^\sharp)$ with $u^\sharp \leqslant \Theta_{k_1}(r)$ if and only if $r > \lambda_{1,k_1}$ and $-c_2 > \lambda_{1,k_3}(-\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$. In addition, if $r - \frac{a_2}{e_2} > \lambda_{1,k_1}$ and $-c_2 > \lambda_{1,k_3}(-\frac{m_2\Theta_{k_1}(r-\frac{a_2}{e_2})}{1+\Theta_{k_1}(r-\frac{a_2}{e_2})})$, then the positive solution $(u^\sharp, v^\sharp)$ satisfies $\Theta_{k_1}(r - \frac{a_2}{e_2}) \leqslant u^\sharp$ and $\tilde{w} \leqslant w^\sharp$, where $\tilde{w}$ is the unique positive solution of the following problem*

$$\begin{cases} -\Delta w = w\left(\dfrac{m_2\Theta_{k_1}(r-\frac{a_2}{e_2})}{1+\Theta_{k_1}(r-\frac{a_2}{e_2})+e_2 w} - c_2\right) & \text{in } \Omega, \\ k_3\dfrac{\partial w}{\partial \nu} + w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

Let $S_1 = \{(u^*, v^*, 0)$: where (u^*, v^*) is the positive solution of (3.9)$\}$ and $S_2 = \{(u^\sharp, 0, w^\sharp)$: where $(u^\sharp, 0, w^\sharp)$ is the positive solution of (3.10)$\}$. Then by using Lemma 2.5, we can get the following lemma.

Lemma 3.10. *Assume that $r > \lambda_{1,k_1}$ and $-c_1 > \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$.*

(i) *If $-c_2 > \lambda_{1,k_3}(-\frac{m_2u^*}{1+u^*})$ for any $(u^*, v^*, 0) \in S_1$, then* $\deg_{\mathbb{W}}(I - \Re, S_1) = 0$.
(ii) *If $-c_2 < \lambda_{1,k_3}(-\frac{m_2u^*}{1+u^*})$ for any $(u^*, v^*, 0) \in S_1$, then* $\deg_{\mathbb{W}}(I - \Re, S_1) = 1$.

Proof. Recalling the definitions of $\mathbb{E}$, $\mathbb{W}$ and $\mathbb{D}$ in Section 3, we define $\mathbb{E}_1 = C^1_{k_1}(\overline{\Omega}) \oplus C^1_{k_2}(\overline{\Omega})$, $\mathbb{E}_2 = C^1_{k_3}(\overline{\Omega})$ and denote $\mathbb{W}_1 = \mathbb{N}_1 \oplus \mathbb{N}_2$, $\mathbb{W}_2 = \mathbb{N}_3$. Define

$$\Re_1(u, v, w) = (-\Delta + q)^{-1}\begin{pmatrix} u(r - u - \frac{a_1 v}{1+u+e_1 v} - \frac{a_2 w}{1+u+e_2 w}) + qu \\ v(\frac{m_1 u}{1+u+e_1 v} - c_1) + qv \end{pmatrix},$$

$$\Re_2(u, v, w) = (-\Delta + q)^{-1}\left(u\left(r - u - \frac{a_1 v}{1+u+e_1 v}\right) + qu\right).$$

Then $\Re = (\Re_1, \Re_2)$. We choose a neighborhood $U \subset \mathbb{W}_1 \cap \mathbb{D}$ of $S_1 \cap \mathbb{W}_1$ such that $(\Theta_{k_1}(r), 0) \notin \overline{U}$. Now, $\Re_1(u, v, 0) = (u, v)$ with $(u, v) \in \overline{U}$ if and only if $(u, v, 0) \in S_1$. Then we are in the framework to use Theorem 2.5.

For any $(u^*, v^*, 0) \in S_1$, we have

$$\Re_2'(u^*, v^*, 0)\big|_{\mathbb{W}_2} w = (-\Delta + q)^{-1}\left(q - c_2 + \frac{m_2 u^*}{1+u^*}\right)w.$$

Notice that $q - c_2 + \frac{m_2u^*}{1+u^*} > 0$ in Ω for any $(u^*, v^*, 0) \in S_1$ by our choice of q. So, by using maximum principle, we can see that $\Re_2'(u^*, v^*, 0)|_{\mathbb{W}_2}$ is u_0-positive in the sense of [46] with $u_0 = (-\Delta)^{-1}1$. Hence $r(\Re_2'(u^*, v^*, 0)|_{\mathbb{W}_2}) > 0$ and it is the only eigenvalue corresponding to a positive eigenvector.

From the definition of $\Re_2'(u^*, v^*, 0)|_{\mathbb{W}_2}$, we can easily show that $r(\Re_2'(u^*, v^*, 0)|_{\mathbb{W}_2}) > 1$ if and only if $-c_2 > \lambda_{1,k_3}(-\frac{m_2u^*}{1+u^*})$ and $r(\Re_2'(u^*, v^*, 0)|_{\mathbb{W}_2}) < 1$ if and only if $-c_2 < \lambda_{1,k_3}(-\frac{m_2u^*}{1+u^*})$. So by Theorem 2.5, we have

$$\deg_{\mathbb{W}}\big(I - \Re, U \times \mathbb{W}_2(\epsilon), 0\big) = \begin{cases} 0 & \text{if } -c_2 > \lambda_{1,k_3}(-\frac{m_2u^*}{1+u^*}), \\ \deg_{\mathbb{W}_1}(I - \Re_1, U, 0) & \text{if } -c_2 < \lambda_{1,k_3}(-\frac{m_2u^*}{1+u^*}). \end{cases} \tag{3.14}$$

On the other hand, following the results in [16,17], we have

$$\deg_{\mathbb{W}_1}(I-\Re_1,U,0)=\begin{cases}1 & \text{if } r>\lambda_{1,k_1} \text{ and } -c_1>\lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}),\\ -1 & \text{if } r<\lambda_{1,k_1} \text{ and } -c_1<\lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}),\\ 0 & \text{if } (r-\lambda_{1,k_1})(-c_1-\lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}))<0.\end{cases} \tag{3.15}$$

Therefore, from (3.14) and (3.15) and the conditions of Lemma 3.10, we can get

$$\deg_{\mathbb{W}}\big(I-\Re,U\times\mathbb{W}_2(\epsilon),0\big)=\begin{cases}0 & \text{if } -c_2>\lambda_{1,k_3}(-\frac{m_2u^*}{1+u^*}),\\ 1 & \text{if } -c_2<\lambda_{1,k_3}(-\frac{m_2u^*}{1+u^*}).\end{cases} \tag{3.16}$$

Since the degree discussed above does not depend on the particular choice of U and ϵ and $S_1\neq\emptyset$ implies $r>\lambda_{1,k_1}$ and $-c_1>\lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$, by using (3.16), we complete the proof. □

Similarly, we can obtain the following lemma by using the similar methods as above.

Lemma 3.11. *Assume that $r>\lambda_{1,k_1}$ and $-c_2>\lambda_{1,k_3}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$.*

(i) *If $-c_1>\lambda_{1,k_2}(-\frac{m_1u^\sharp}{1+u^\sharp})$ for any $(u^\sharp,0,w^\sharp)\in S_2$, then $\deg_{\mathbb{W}}(I-\Re,S_2)=0$.*
(ii) *If $-c_1<\lambda_{1,k_2}(-\frac{m_2u^\sharp}{1+u^\sharp})$ for any $(u^\sharp,0,w^\sharp)\in S_2$, then $\deg_{\mathbb{W}}(I-\Re,S_2)=1$.*

Based on above analysis, we can give the following results about the existence of coexistence states of (1.2).

Theorem 3.12. *If $r-\frac{a_1}{e_1}>\lambda_{1,k_1}$, $r-\frac{a_2}{e_2}>\lambda_{1,k_1}$, $\lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r-\frac{a_1}{e_1})}{1+\Theta_{k_1}(r-\frac{a_1}{e_1})})<-c_1<\lambda_{1,k_2}$ and $\lambda_{1,k_3}(-\frac{m_2\Theta_{k_1}(r-\frac{a_2}{e_2})}{1+\Theta_{k_1}(r-\frac{a_2}{e_2})})<-c_2<\lambda_{1,k_3}$, then (1.2) has at least one coexistence state.*

Proof. Since $r-\frac{a_1}{e_1}>\lambda_{1,k_2}$ and $r-\frac{a_2}{e_2}>\lambda_{1,k_3}$, we can obtain $\deg_{\mathbb{W}}(I-\Re,\mathbb{D})=1$ and $\mathrm{index}_{\mathbb{W}}(\Re,(0,0,0))=0$ from Lemma 3.6. Thus, it suffices to show that

$$\mathrm{index}_{\mathbb{W}}\big(\Re,\big(\Theta_{k_1}(r),0,0\big)\big)+\deg_{\mathbb{W}}(I-\Re,S_1)+\deg_{\mathbb{W}}(I-\Re,S_2)\neq 1.$$

Since

$$-c_1>\lambda_{1,k_2}\left(-\frac{m_1\Theta_{k_1}(r-\frac{a_1}{e_1})}{1+\Theta_{k_1}(r-\frac{a_1}{e_1})}\right)>\lambda_{1,k_2}\left(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\right)$$

and

$$-c_2>\lambda_{1,k_3}\left(-\frac{m_2\Theta_{k_1}(r-\frac{a_2}{e_2})}{1+\Theta_{k_1}(r-\frac{a_2}{e_2})}\right)>\lambda_{1,k_3}\left(-\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\right),$$

we have $\mathrm{index}_{\mathbb{W}}(\Re,(\Theta_{k_1}(r),0,0))=0$ from Lemma 3.7. Moreover, noting that $\Theta_{k_1}(r-\frac{a_1}{e_1})\leqslant u^*$ and $\Theta_{k_1}(r-\frac{a_2}{e_2})\leqslant u^\sharp$, from Theorems 3.8 and 3.9, we have

$$-c_1>\lambda_{1,k_2}\left(-\frac{m_1\Theta_{k_1}(r-\frac{a_1}{e_1})}{1+\Theta_{k_1}(r-\frac{a_1}{e_1})}\right)>\lambda_{1,k_2}\left(-\frac{m_1u^*}{1+u^*}\right)$$

and

$$-c_2>\lambda_{1,k_3}\left(-\frac{m_2\Theta_{k_1}(r-\frac{a_2}{e_2})}{1+\Theta_{k_1}(r-\frac{a_2}{e_2})}\right)>\lambda_{1,k_3}\left(-\frac{m_2u^\sharp}{1+u^\sharp}\right)$$

by the comparison principle of principle eigenvalue. So we have $\deg_{\mathbb{W}}(I-\Re,S_1)=0$ and $\deg_{\mathbb{W}}(I-\mathbb{W},S_2)=0$ from Lemmas 3.10 and 3.11. Therefore, $\mathrm{index}_{\mathbb{W}}(\Re,(\Theta_{k_1(r)},0,0))+\deg_{\mathbb{W}}(I-\Re,S_1)+\deg_{\mathbb{W}}(I-\Re,S_2)=0\neq 1$. The proof is completed. □

4. Stability and uniqueness of coexistence states

In this section, we investigate the stability and uniqueness of coexistence states of (1.2). In order to give our main results, we introduce some notations at first.

Let $u^\dagger = \Theta_{k_1}(r)$ be the unique positive solution of the following system

$$\begin{cases} -\Delta u = u(r-u) & \text{in } \Omega, \\ k_1 \dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

when $r > \lambda_{1,k_1}$. Denote by $v^\dagger$ the unique positive solution of the following system

$$\begin{cases} -\Delta v = v\left(\dfrac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)+e_1 v} - c_1\right) & \text{in } \Omega, \\ k_2 \dfrac{\partial v}{\partial \nu} + v = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

if $-c_1 > \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$. Let $w^\dagger$ denote the unique positive solution of the following system

$$\begin{cases} -\Delta w = w\left(\dfrac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)+e_2 w} - c_2\right) & \text{in } \Omega, \\ k_3 \dfrac{\partial w}{\partial \nu} + w = 0 & \text{on } \partial\Omega \end{cases} \tag{4.3}$$

if $-c_2 > \lambda_{1,k_3}(-\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$.

The following theorem shows the uniqueness, non-degeneracy and linear stability of the coexistence states for (1.2) under some assumptions.

Theorem 4.1. *Assume that the conditions in Theorem* 3.12 *hold.*

(i) *The coexistence states of* (1.2) *converge to* $(u^\dagger, v^\dagger, w^\dagger)$ *as* $a_i \to 0$ *for* $i = 1, 2$.
(ii) *There exists a positive constant* $\tilde{a} = \tilde{a}(a_1, a_2)$ *such that for* $a_1, a_2 < \tilde{a}$, (1.2) *has exactly one coexistence state which is non-degenerate and linearly stable.*
(iii) *The coexistence states of* (1.2) *converge to* $(u^\dagger, 0, 0)$ *as* $e_i \to \infty$ *for* $i = 1, 2$.
(iv) *There exists a positive constant* $\tilde{e} = \tilde{e}(e_1, e_2)$ *such that for* $e_1, e_2 > \tilde{e}$, (1.2) *has exactly one coexistence state which is non-degenerate and linearly stable.*

Proof. (i) It is easy to see that the compact operator $\Re(u, v, w)$ defined in Section 3 converges to the operator

$$\widehat{\Re}(u, v, w) = (-\Delta + q)^{-1}\begin{pmatrix} u(r-u) + qu \\ v(\frac{m_1}{1+u+e_1 v} - c_1) + qv \\ w(\frac{m_2 u}{1+u+e_2 w} - c_2) + qw \end{pmatrix}$$

as $a_i \to 0$ for $i = 1, 2$. So the fixed points of (1.2) converge to the fixed points of $\widehat{\Re}(u, v, w)$. Noting that $(u^\dagger, v^\dagger, w^\dagger)$ is the only fixed point of $\widehat{\Re}(u, v, w)$, the conclusion follows.

(ii) At first, we shall prove that the coexistence state is non-degenerate and linearly stable. In view of [45, Theorem 11.20], it is sufficient to show that the corresponding linearized eigenvalue problem of (1.2) has no eigenvalue λ with $\mathrm{Re}(\lambda) \leqslant 0$. We argue by contradiction. Suppose that (1.2) has a coexistence state (u_i, v_i, w_i) which is either degenerate or linearly unstable for sequences $\{a_{1,i}\}$ and $\{a_{2,i}\}$ with $a_{1,i}, a_{2,i} \to 0$, where $i \geqslant 1$. So there exist λ_i with $\mathrm{Re}(\lambda_i) \leqslant 0$ and $(\xi_i, \zeta_i, \eta_i) \neq (0, 0, 0)$ such that

$$\begin{cases} -\Delta\xi_i - \left(r - 2u_i - \dfrac{a_1 v_i(1+e_1 v_i)}{(1+u_i+e_1 v_i)^2} - \dfrac{a_2 w_i(1+e_1 w_i)}{(1+u_i+e_2 w_i)^2}\right)\xi_i \\ \quad + \dfrac{a_1 u_i(1+u_i)}{(1+u_i+e_1 v_i)^2}\zeta_i + \dfrac{a_2 u_i(1+u_i)}{(1+u_i+e_2 w_i)^2}\eta_i = \lambda_i \xi_i, \\ -\Delta\zeta_i - \dfrac{m_1 v_i(1+e_1 v_i)}{(1+u_i+e_1 v_i)^2}\xi_i - \left(\dfrac{m_1 u_i(1+u_i)}{(1+u_i+e_1 v_i)^2} - c_1\right)\zeta_i = \lambda_i \zeta_i, \\ -\Delta\eta_i - \dfrac{m_2 w_i(1+e_2 w_i)}{(1+u_i+e_2 w_i)^2}\xi_i - \left(\dfrac{m_2 u_i(1+u_i)}{(1+u_i+e_2 w_i)^2} - c_1\right)\eta_i = \lambda_i \eta_i & \text{in } \Omega, \\ k_1 \dfrac{\partial \xi_i}{\partial \nu} + \xi_i = k_2 \dfrac{\partial \zeta_i}{\partial \nu} + \zeta_i = k_3 \dfrac{\partial \eta_i}{\partial \nu} + \eta_i = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Assume that $\|\xi_i\|_{L^2}^2 + \|\zeta_i\|_{L^2}^2 + \|\eta_i\|_{L^2}^2 = 1$. Then from the system (4.4), by a simple calculation, we can obtain

$$\begin{aligned}
\lambda_i = {} & \int_\Omega |\nabla\xi_i|^2\,dx - \int_\Omega \left(r - 2u_i - \frac{a_1 v_i(1+e_1 v_i)}{(1+u_i+e_1 v_i)^2} - \frac{a_2 w_i(1+e_1 w_i)}{(1+u_i+e_2 w_i)^2}\right)|\xi_i|^2\,dx \\
& + \int_\Omega \frac{a_1 u_i(1+u_i)}{(1+u_i+e_1 v_i)^2}\zeta_i\overline{\xi_i}\,dx + \int_\Omega \frac{a_2 u_i(1+u_i)}{(1+u_i+e_2 w_i)^2}\eta_i\overline{\xi_i}\,dx + \tau_1 \int_{\partial\Omega} |\nabla\xi_i|^2\,dx \\
& + \int_\Omega |\nabla\zeta_i|^2\,dx - \int_\Omega \frac{m_1 v_i(1+e_1 v_i)}{(1+u_i+e_1 v_i)^2}\xi_i\overline{\zeta_i}\,dx - \int_\Omega \left(\frac{m_1 u_i(1+u_i)}{(1+u_i+e_1 v_i)^2} - c_1\right)|\zeta_i|^2\,dx + \tau_2 \int_{\partial\Omega} |\nabla\zeta_i|^2\,dx \\
& + \int_\Omega |\nabla\eta_i|^2\,dx - \int_\Omega \frac{m_2 w_i(1+e_2 w_i)}{(1+u_i+e_2 w_i)^2}\xi_i\overline{\eta_i}\,dx - \int_\Omega \left(\frac{m_2 u_i(1+u_i)}{(1+u_i+e_2 w_i)^2} - c_1\right)|\eta_i|^2\,dx + \tau_3 \int_{\partial\Omega} |\nabla\eta_i|^2\,dx,
\end{aligned}$$

where $\overline{\xi_i}$, $\overline{\zeta_i}$ and $\overline{\eta_i}$ are the respective complex conjugates of ξ_i, ζ_i and η_i. Moreover, τ_i is defined by $\frac{1}{k_i}$ for $k_i > 0$ and 0 for $k_i = 0$. It is easy to show that $\operatorname{Im}\lambda_i$ and $\operatorname{Re}\lambda_i$ are bounded, and hence λ_i is bounded. So we may assume that $\lambda_i \to \lambda$ and then $\operatorname{Re}\lambda \leqslant 0$. By L^p estimate, we have $\|\xi_i\|_{W^{2,2}}$, $\|\zeta_i\|_{W^{2,2}}$ and $\|\eta_i\|_{W^{2,2}}$ are bounded. Hence we may assume that $\xi_i \to \xi$, $\zeta_i \to \zeta$ and $\eta_i \to \eta$ in H^1 strongly. Taking the limit in (4.4), we obtain

$$\begin{cases}
-\Delta\xi - (r - 2u^\dagger)\xi = \lambda\xi, \\
-\Delta\zeta - \dfrac{m_1 v^\dagger(1+e_1 v^\dagger)}{(1+u^\dagger+e_1 v^\dagger)^2}\xi - \left(\dfrac{m_1 u^\dagger(1+u^\dagger)}{(1+u^\dagger+e_1 v^\dagger)^2} - c_1\right)\zeta = \lambda\zeta, \\
-\Delta\eta - \dfrac{m_2 w^\dagger(1+e_2 w^\dagger)}{(1+u^\dagger+e_2 w^\dagger)^2}\xi - \left(\dfrac{m_2 u^\dagger(1+u^\dagger)}{(1+u^\dagger+e_2 w^\dagger)^2} - c_1\right)\eta = \lambda\eta & \text{in } \Omega, \\
k_1\dfrac{\partial\xi}{\partial\nu} + \xi = k_2\dfrac{\partial\zeta}{\partial\nu} + \zeta = k_3\dfrac{\partial\eta}{\partial\nu} + \eta = 0 & \text{on } \partial\Omega.
\end{cases} \tag{4.5}$$

Obviously $\lambda \in \mathbb{R}$. If $\xi \neq 0$, then $\lambda = \lambda_{1,k_1}(2u^\dagger - r) = \lambda_{1,k_1}(2\Theta_{k_1}(r) - r) > \lambda_{1,k_1}(\Theta_{k_1}(r) - r) = 0$. However, $\operatorname{Re}\lambda \leqslant 0$, which is a contradiction. Hence $\xi = 0$. Then from the last two equations of (4.5), we have

$$\begin{cases}
-\Delta\zeta - \left(\dfrac{m_1 u^\dagger(1+u^\dagger)}{(1+u^\dagger+e_1 v^\dagger)^2} - c_1\right)\zeta = \lambda\zeta, \\
-\Delta\eta - \left(\dfrac{m_2 u^\dagger(1+u^\dagger)}{(1+u^\dagger+e_2 w^\dagger)^2} - c_1\right)\eta = \lambda\eta & \text{in } \Omega, \\
k_2\dfrac{\partial\zeta}{\partial\nu} + \zeta = k_3\dfrac{\partial\eta}{\partial\nu} + \eta = 0 & \text{on } \partial\Omega.
\end{cases} \tag{4.6}$$

If $\zeta \neq 0$, then $\lambda = \lambda_{1,k_2}(c_1 - \frac{m_1 u^\dagger(1+u^\dagger)}{(1+u^\dagger+e_1 v^\dagger)^2}) > \lambda_{1,k_2}(c_1 - \frac{m_1 u^\dagger}{(1+u^\dagger+e_1 v^\dagger)}) = 0$, which is a contradiction. Hence $\zeta = 0$. Similarly, we can show that $\eta = 0$. This contradiction indicates that the coexistence state of (1.2) is non-degenerate and linearly stable.

Next, we prove the uniqueness of coexistence state of (1.2). By compactness, $\Re$ has at most finitely many positive fixed points in the region $\mathbb{N}$ defined in Section 3 and denote them by (u_i, v_i, w_i) for $i = 1, \dots, k$. For sufficiently small a_1, a_2, as in the proof of Lemma 3.7, it is easy to show that $I - \Re'(u_i, v_i, w_i)$ is invertible and $\Re'(u_i, v_i, w_i)$ has no property α on $\overline{\mathbb{W}}_{(u_i,v_i,w_i)}$. In addition, $\Re'(u_i, v_i, w_i)$ does not have a real eigenvalue which is greater than or equal to 1. Thus from Theorem 2.4, we have $\operatorname{index}(\Re, (u_i, v_i, w_i)) = (-1)^0 = 1$ for $i = 1, \dots, k$. Then by using the additivity property of the degree, we have

$$\begin{aligned}
k = {} & \sum_{i=1}^{k} \operatorname{index}_{\mathbb{W}}\big(\Re, (u_i, v_i, w_i)\big) = \deg_{\mathbb{W}}(I - \Re, \mathbb{D}) - \operatorname{index}_{\mathbb{W}}\big(\Re, (0,0,0)\big) \\
& - \operatorname{index}_{\mathbb{W}}\big(\Re, \big(\Theta_{k_1}(r), 0, 0\big)\big) - \deg_{\mathbb{W}}(I - \Re, S_1) - \deg_{\mathbb{W}}(I - \Re, S_2) \\
= {} & 1 - 0 - 0 - 0 - 0 = 1.
\end{aligned}$$

The uniqueness is obtained.

(iii) It is easy to see that the compact operator $\Re(u, v, w)$ defined in Section 3 converges to the operator

$$\widetilde{\Re}(u, v, w) = (-\Delta + q)^{-1}\begin{pmatrix} u(r-u) + qu \\ -c_1 v + qv \\ -c_2 w + qw \end{pmatrix}$$

as $e_i \to \infty$ for $i = 1, 2$. Hence the fixed points of (1.2) converge to the fixed points of $\widehat{\Re}(u, v, w)$. Since $(u^*, 0, 0)$ is the only fixed point of $\widehat{\Re}(u, v, w)$, the conclusion follows.

(iv) As in the proof of (ii), we can derive a contradiction to prove that the coexistence state is non-degenerate and linearly stable and get the uniqueness by using the additivity property of the degree. So we omit the proof. □

5. Asymptotic behavior: Extinction and global attractor

In this section, we shall investigate the asymptotic behavior of the time-dependent solutions of system (1.1). First, we give the sufficient conditions for the extinction and permanence to system (1.1).

Theorem 5.1. *Let (u, v, w) be a positive solution of (1.1).*

(i) *If $r \leqslant \lambda_{1,k_1}$, then $(u, v, w) \to (0, 0, 0)$ as $t \to \infty$.*
(ii) *If $r > \lambda_{1,k_1}$, $-c_1 \leqslant \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$ and $-c_2 \leqslant \lambda_{1,k_3}(-\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$, then $(u, v, w) \to (\Theta_{k_1}(r), 0, 0)$ as $t \to \infty$.*

Proof. (i) First, it is easy to see that any time-dependent solution (u, v, w) of (1.1) satisfies

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u \leqslant u(r-u) & \text{in } \Omega \times (0, \infty), \\ k_1 \dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{5.1}$$

Since $r \leqslant \lambda_{1,k_1}$, it follows from Theorem 2.3 and comparison principle that $u \to 0$ as $t \to \infty$ uniformly. Let ϵ be a sufficiently small positive constant such that $\epsilon < \min\{\frac{c_1}{m_1}, \frac{c_2}{m_2}\}$. Then, there exists a $T(\epsilon)$ such that $u(x,t) \leqslant \epsilon$ for all $t > T(\epsilon)$. So we have

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v \leqslant v(m_1\epsilon - c_1) < 0 & \text{in } \Omega \times \big(T(\epsilon), \infty\big), \\ k_2 \dfrac{\partial v}{\partial \nu} + v = 0 & \text{on } \partial\Omega \times \big(T(\epsilon), \infty\big), \end{cases} \tag{5.2}$$

which implies that $v \to 0$ as $t \to \infty$ uniformly by Theorem 2.3 and comparison principle again. Similarly, we can show that $w \to 0$ uniformly as $t \to \infty$.

(ii) Since $r > \lambda_{1,k_1}$, it follows from (5.1), Theorem 2.3 and comparison principle that

$$\limsup_{t\to\infty} u(x,t) \leqslant \Theta_{k_1}(r). \tag{5.3}$$

Let ϵ be a sufficiently small positive constant which satisfies the following two conditions:

$$\text{(a)} \quad \epsilon < \frac{\lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}) + c_1}{m_1}, \qquad \text{(b)} \quad \epsilon < \frac{r - \lambda_{1,k_1}}{a_1 + a_2}.$$

Then, there exists a $T(\epsilon) \geqslant 0$ such that $u(x,t) \leqslant \Theta_{k_1}(r) + \epsilon$ for all $t > T(\epsilon)$. Therefore, we have

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v \leqslant v\left(\dfrac{m_1(\Theta_{k_1}(r)+\epsilon)}{1+\Theta_{k_1}(r)+\epsilon+e_1 v} - c_1\right) \leqslant v\left(\dfrac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)+e_1 v} + m_1\epsilon - c_1\right) & \text{in } \big(T(\epsilon), \infty\big) \times \Omega, \\ k_2 \dfrac{\partial v}{\partial \nu} + v = 0 & \text{on } \big(T(\epsilon), \infty\big) \times \partial\Omega. \end{cases} \tag{5.4}$$

Since ϵ satisfies (a) and $-c_1 \leqslant \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$, we conclude that $v \to 0$ uniformly as $t \to \infty$ by Theorem 2.3 and comparison principle. Similarly, we can show that $w \to 0$ uniformly as $t \to \infty$. Then, there exists a $T'(\epsilon) \geqslant 0$ such that $v(x,t), w(x,t) < \epsilon$ for all $t > T'(\epsilon)$. So we have

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u \geqslant u(r - u - a_1\epsilon - a_2\epsilon) & \text{in } \Omega \times \big(T'(\epsilon), \infty\big), \\ k_1 \dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial\Omega \times \big(T'(\epsilon), \infty\big). \end{cases} \tag{5.5}$$

Since ϵ satisfies (b), by Theorem 2.3 and comparison principle, we have

$$\liminf_{t\to\infty} u(x,t) \geqslant \Theta_{k_1}(r - a_1\epsilon - a_2\epsilon). \tag{5.6}$$

So, by (5.3) and (5.6), we have

$$\Theta_{k_1}(r - a_1\epsilon - a_2\epsilon) \leqslant \liminf_{t\to\infty} u(x,t) \leqslant \limsup_{t\to\infty} u(x,t) \leqslant \Theta_{k_1}(r). \tag{5.7}$$

By the continuity for $\epsilon \to 0$, we conclude $u(x,t) \to \Theta_{k_1}(r)$ uniformly as $t \to \infty$ from (5.7). This completes the proof. □

Definition 5.2. A pair of functions $(\overline{u}, \overline{v}, \overline{w})$ and $(\underline{u}, \underline{v}, \underline{w})$ in $C(\overline{\Omega}) \cap C^2(\Omega)$ are called ordered upper and lower solution of (1.2) if they satisfy the relation $\overline{u} \geqslant \underline{u}$, $\overline{v} \geqslant \underline{v}$, $\overline{w} \geqslant \underline{w}$ and the following inequalities:

$$\begin{cases} -\Delta\overline{u} \geqslant \overline{u}(r-\overline{u}) - \dfrac{a_1\overline{u}\underline{v}}{1+\overline{u}+e_1\underline{v}} - \dfrac{a_2\overline{u}\underline{w}}{1+\overline{u}+e_2\underline{w}}, \\ -\Delta\underline{u} \leqslant \underline{u}(r-\underline{u}) - \dfrac{a_1\underline{u}\overline{v}}{1+\underline{u}+e_1\overline{v}} - \dfrac{a_2\underline{u}\overline{w}}{1+\underline{u}+e_1\overline{w}}, \\ -\Delta\overline{v} \geqslant \dfrac{m_1\overline{u}\overline{v}}{1+\overline{u}+e_1\overline{v}} - c_1\overline{v}, \\ -\Delta\underline{v} \leqslant \dfrac{m_1\underline{u}\underline{v}}{1+\underline{v}+e_1\underline{v}} - c_1\underline{v}, \\ -\Delta\overline{w} \geqslant \dfrac{m_2\overline{u}\overline{w}}{1+\overline{u}+e_2\overline{w}} - c_2\overline{w}, \\ -\Delta\underline{w} \leqslant \dfrac{m_2\underline{u}\underline{w}}{1+\underline{w}+e_2\underline{v}} - c_2\underline{w} & \text{in } \Omega, \\ k_1\dfrac{\partial\overline{u}}{\partial\nu} + \overline{u} \geqslant 0 \geqslant k_1\dfrac{\partial\underline{u}}{\partial\nu} + \underline{u}, \\ k_2\dfrac{\partial\overline{v}}{\partial\nu} + \overline{v} \geqslant 0 \geqslant k_2\dfrac{\partial\underline{v}}{\partial\nu} + \underline{v}, \\ k_3\dfrac{\partial\overline{w}}{\partial\nu} + \overline{w} \geqslant 0 \geqslant k_3\dfrac{\partial\underline{w}}{\partial\nu} + \underline{w} & \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

In order to give the main results, we introduce the following assumptions:

$$\begin{cases} r > \dfrac{a_1}{e_1} + \dfrac{a_2}{e_2}, \\ -c_1 > \lambda_{1,k_2}\left(-\dfrac{m_1\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})}{1+\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})}\right), \\ -c_2 > \lambda_{1,k_3}\left(-\dfrac{m_2\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})}{1+\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})}\right). \end{cases} \tag{5.9}$$

Let $v^{\triangledown}$ be the unique positive solution of the following problem

$$\begin{cases} -\Delta v = v\left(\dfrac{m_1\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})}{1+\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})+e_1 v} - c_1\right) & \text{in } \Omega, \\ k_2\dfrac{\partial v}{\partial\nu} + v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.10}$$

and $v^{\vartriangle}$ be the unique positive solution of the following problem

$$\begin{cases} -\Delta v = v\left(\dfrac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)+e_1 v} - c_1\right) & \text{in } \Omega, \\ k_2\dfrac{\partial v}{\partial\nu} + v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.11}$$

Let $w^{\triangledown}$ be the unique positive solution of the following problem

$$\begin{cases} -\Delta w = w\left(\dfrac{m_2\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})}{1+\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})+e_2 w} - c_2\right) & \text{in } \Omega, \\ k_3\dfrac{\partial w}{\partial\nu} + w = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.12}$$

and $w^{\triangle}$ be the unique positive solution of the following problem

$$\begin{cases} -\Delta w = w\left(\dfrac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)+e_2 w} - c_2\right) & \text{in } \Omega, \\ k_3\dfrac{\partial w}{\partial \nu} + w = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.13}$$

Remark 5.3. By Theorem 2.3, it is easy to see that the existence and uniqueness of v^{∇}, $v^{\triangle}$, w^{∇} and $w^{\triangle}$ follow from the assumptions (5.9).

The following theorem provides sufficient conditions for permanence of the time-dependent system (1.1).

Theorem 5.4. *Assume that the conditions in* (5.9) *hold. Then, there exist a pair of functions* $(\tilde{u}, \tilde{v}, \tilde{w})$ *and* $(\hat{u}, \hat{v}, \hat{w})$ *in* $C(\overline{\Omega}) \cap C^2(\Omega)$ *such that*

$$\begin{cases} -\Delta \tilde{u} = \tilde{u}(r-\tilde{u}) - \dfrac{a_1\tilde{u}\hat{v}}{1+\tilde{u}+e_1\hat{v}} - \dfrac{a_2\tilde{u}\hat{w}}{1+\tilde{u}+e_2\hat{w}}, \\ -\Delta \hat{u} = \hat{u}(r-\hat{u}) - \dfrac{a_1\hat{u}\tilde{v}}{1+\hat{u}+e_1\tilde{v}} - \dfrac{a_2\hat{u}\tilde{w}}{1+\hat{u}+e_1\tilde{w}}, \\ -\Delta \tilde{v} = \dfrac{m_1\tilde{u}\tilde{v}}{1+\tilde{u}+e_1\tilde{v}} - c_1\tilde{v}, \\ -\Delta \hat{v} = \dfrac{m_1\hat{u}\hat{v}}{1+\hat{u}+e_1\hat{v}} - c_1\hat{v}, \\ -\Delta \tilde{w} = \dfrac{m_2\tilde{u}\tilde{w}}{1+\tilde{u}+e_2\tilde{w}} - c_2\tilde{w}, \\ -\Delta \hat{w} = \dfrac{m_2\hat{u}\hat{w}}{1+\hat{u}+e_2\hat{w}} - c_2\hat{w} & \textit{in } \Omega, \\ k_1\dfrac{\partial \tilde{u}}{\partial \nu} + \tilde{u} = 0 = k_1\dfrac{\partial \hat{u}}{\partial \nu} + \hat{u}, \\ k_2\dfrac{\partial \tilde{v}}{\partial \nu} + \tilde{v} = 0 = k_2\dfrac{\partial \hat{v}}{\partial \nu} + \hat{v}, \\ k_3\dfrac{\partial \tilde{w}}{\partial \nu} + \tilde{w} = 0 = k_3\dfrac{\partial \hat{w}}{\partial \nu} + \hat{w} & \textit{on } \partial\Omega, \end{cases} \tag{5.14}$$

and satisfy the following relations $\Theta_{k_1}(r - \frac{a_1}{e_1} - \frac{a_2}{e_2}) \leqslant \hat{u} \leqslant \tilde{u} \leqslant \Theta_{k_1}(r)$, $v^{\nabla} \leqslant \hat{v} \leqslant \tilde{v} \leqslant v^{\triangle}$, $w^{\nabla} \leqslant \hat{w} \leqslant \tilde{w} \leqslant w^{\triangle}$. *Furthermore,* $[\hat{u}, \tilde{u}] \times [\hat{v}, \tilde{v}] \times [\hat{w}, \tilde{w}]$ *is a positive global attractor of* (1.1).

Remark 5.5. We point out such functions $(\tilde{u}, \tilde{v}, \tilde{w})$ and $(\hat{u}, \hat{v}, \hat{w})$ are called quasisolution of (1.2).

Proof of Theorem 5.4. It is easy to show that $(\Theta_{k_1}(r), v^{\triangle}, w^{\triangle})$ and $(\Theta_{k_1}(r - \frac{a_1}{e_1} - \frac{a_2}{e_2}), v^{\nabla}, w^{\nabla})$ are a pair of ordered upper and lower solution of (1.1) under assumption (5.9). So the existence of a pair of functions $(\tilde{u}, \tilde{v}, \tilde{w})$ and $(\hat{u}, \hat{v}, \hat{w})$ can be proved by the iteration scheme in [47] easily.

Next, we prove that $[\hat{u}, \tilde{u}] \times [\hat{v}, \tilde{v}] \times [\hat{w}, \tilde{w}]$ is a positive global attractor of (1.1). Noting that $(\hat{u}, \hat{v}, \hat{w})$ is positive in Ω by the maximum principle, it suffices to prove that $[\hat{u}, \tilde{u}] \times [\hat{v}, \tilde{v}] \times [\hat{w}, \tilde{w}]$ is a global attractor.

Let ϵ be sufficiently small such that

$$\text{(a)}\quad \epsilon < \frac{-\lambda_{1,k_2}\left(-\frac{m_1\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})}{1+\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})}\right) - c_1}{m_1} < \frac{-\lambda_{1,k_2}\left(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\right) - c_1}{m_1};$$

$$\text{(b)}\quad \epsilon < \frac{-\lambda_{1,k_3}\left(-\frac{m_2\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})}{1+\Theta_{k_1}(r-\frac{a_1}{e_1}-\frac{a_2}{e_2})}\right) - c_2}{m_2} < \frac{-\lambda_{1,k_3}\left(-\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)}\right) - c_2}{m_2}.$$

Since

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u \leqslant u(r-u) & \text{in } (0,\infty)\times\Omega, \\ k_1\dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } (0,\infty)\times\partial\Omega, \end{cases} \tag{5.15}$$

it follows from Theorem 2.3, comparison principle and assumption (5.9) that $\limsup_{t\to\infty} u(x,t) \leqslant \Theta_{k_1}(r)$. So, there exists a $T(\epsilon) \geqslant 0$ such that

$$u(x,t) \leqslant \Theta_{k_1}(r) + \epsilon \quad \text{for all } t > T(\epsilon). \tag{5.16}$$

Then, we get from the second and third equations of (1.1) that

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v \leqslant v\left(\dfrac{m_1(\Theta_{k_1}(r)+\epsilon)}{1+\Theta_{k_1}(r)+\epsilon+e_1 v} - c_1\right) \leqslant v\left(\dfrac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)+e_1 v} + m_1\epsilon - c_1\right) & \text{in } \Omega \times \big(T(\epsilon),\infty\big), \\ k_2\dfrac{\partial v}{\partial \nu} + v = 0 & \text{on } \partial\Omega \times \big(T(\epsilon),\infty\big), \end{cases} \tag{5.17}$$

and

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w \leqslant w\left(\dfrac{m_2(\Theta_{k_1}(r)+\epsilon)}{1+\Theta_{k_1}(r)+\epsilon+e_2 w} - c_2\right) \leqslant w\left(\dfrac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)+e_2 w} + m_2\epsilon - c_2\right) & \text{in } \Omega \times \big(T(\epsilon),\infty\big), \\ k_3\dfrac{\partial w}{\partial \nu} + w = 0 & \text{on } \partial\Omega \times \big(T(\epsilon),\infty\big). \end{cases} \tag{5.18}$$

Since ϵ satisfies (a) and (b), by Theorem 2.3, comparison principle and assumption (5.9), we have $\limsup_{t\to\infty} v(x,t) \leqslant v^{\vartriangle}$ and $\limsup_{t\to\infty} w(x,t) \leqslant w^{\vartriangle}$. Thus, there exists a $T'(\epsilon) \geqslant 0$ such that

$$v(x,t) \leqslant v^{\vartriangle} + \epsilon, \qquad w(x,t) \leqslant w^{\vartriangle} + \epsilon \quad \text{for all } t > T'(\epsilon). \tag{5.19}$$

On the other hand, from the first equation of (1.1), we have

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u \geqslant u\left(r - u - \dfrac{a_1}{e_1} - \dfrac{a_2}{e_2}\right) & \text{in } \Omega \times (0,\infty), \\ k_1\dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{5.20}$$

So, we can obtain $\liminf_{t\to\infty} u(x,t) \geqslant \Theta_{k_1}(r - \frac{a_1}{e_1} - \frac{a_2}{e_2})$. Then, there exists a $T''(\epsilon) \geqslant 0$ such that

$$u(x,t) \geqslant \Theta_{k_1}\left(r - \frac{a_1}{e_1} - \frac{a_2}{e_2}\right) - \epsilon \quad \text{for all } t > T''(\epsilon). \tag{5.21}$$

Therefore, we get from the second and third equations of (1.1) that

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v \geqslant v\left(\dfrac{m_1(\Theta_{k_1}(r - \frac{a_1}{e_1} - \frac{a_2}{e_2}) - \epsilon)}{1+\Theta_{k_1}(r - \frac{a_1}{e_1} - \frac{a_2}{e_2}) - \epsilon + e_1 v} - c_1\right) \\ \qquad\qquad \geqslant v\left(\dfrac{m_1\Theta_{k_1}(r - \frac{a_1}{e_1} - \frac{a_2}{e_2})}{1+\Theta_{k_1}(r - \frac{a_1}{e_1} - \frac{a_2}{e_2}) + e_1 v} - m_1\epsilon - c_1\right) & \text{in } \Omega \times \big(T''(\epsilon),\infty\big), \\ k_2\dfrac{\partial v}{\partial \nu} + v = 0 & \text{on } \partial\Omega \times \big(T''(\epsilon),\infty\big), \end{cases} \tag{5.22}$$

and

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w \geqslant w\left(\dfrac{m_2(\Theta_{k_1}(r - \frac{a_1}{e_1} - \frac{a_2}{e_2}) - \epsilon)}{1+\Theta_{k_1}(r - \frac{a_1}{e_1} - \frac{a_2}{e_2}) - \epsilon + e_2 w} - c_2\right) \\ \qquad\qquad \geqslant w\left(\dfrac{m_2\Theta_{k_1}(r - \frac{a_1}{e_1} - \frac{a_2}{e_2})}{1+\Theta_{k_1}(r - \frac{a_1}{e_1} - \frac{a_2}{e_2}) + e_2 w} - m_2\epsilon - c_2\right) & \text{in } \Omega \times \big(T''(\epsilon),\infty\big), \\ k_3\dfrac{\partial w}{\partial \nu} + w = 0 & \text{on } \partial\Omega \times \big(T''(\epsilon),\infty\big). \end{cases} \tag{5.23}$$

Since ϵ satisfies (a) and (b), we have $\liminf_{t\to\infty} v(x,t) \geqslant v^{\triangledown}$ and $\liminf_{t\to\infty} w(x,t) \geqslant w^{\triangledown}$ by Theorem 2.3, assumption (5.9), and comparison principle. So, there exists a $T'''(\epsilon) \geqslant 0$ such that

$$v(x,t) \geqslant v^{\triangledown} - \epsilon \quad \text{and} \quad w(x,t) \geqslant w^{\triangledown} - \epsilon \quad \text{for all } t > T'''(\epsilon). \tag{5.24}$$

Finally, by (5.16), (5.19), (5.21) and (5.24), we conclude that there exist

$$\overline{T} = \max\big\{T(\varepsilon), T'(\varepsilon), T''(\varepsilon), T'''(\varepsilon)\big\}$$

such that for any nontrivial initial condition $(u_0(x), v_0(x), w_0(x))$, the time-dependent solution (u, v, w) of (1.1) satisfies

$$(u, v, w) \in \left[\Theta_{k_1}(r - a_1/e_1 - a_2/e_2) - \epsilon, \Theta_{k_1}(r) + \epsilon\right] \times \left[v^{\nabla} - \epsilon, v^{\triangle} + \epsilon\right] \times \left[w^{\nabla} - \epsilon, w^{\triangle} + \epsilon\right]$$

for all $t > \overline{T}$. Then, by Corollary 2.1 and Theorem 2.1 in [48], we complete the proof. □

The next final theorem gives sufficient conditions for a global attractor in the case that exactly one species is dying out. This can be proved similarly as in the above theorem, so we omit the proof.

Theorem 5.6.

(i) *If* $r > \lambda_{1k_1} + \frac{a_1}{e_1} + \frac{a_2}{e_2}$, $-c_1 > \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$ *and* $-c_2 < \lambda_{1,k_3}(-\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$, *then there exists a pair of quasisolution* $[\tilde{u}, \tilde{v}]$ *and* $[\hat{u}, \hat{v}]$ *of* (3.9) *with* $\tilde{u} \geqslant \hat{u}$ *and* $\tilde{v} \geqslant \hat{v}$. *Moreover,* $[\hat{u}, \tilde{u}] \times [\hat{v}, \tilde{v}] \times \{0\}$ *is a global attractor of* (1.1).

(ii) *If* $r > \lambda_{1k_1} + \frac{a_1}{e_1} + \frac{a_2}{e_2}$, $-c_1 < \lambda_{1,k_2}(-\frac{m_1\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$ *and* $-c_2 > \lambda_{1,k_3}(-\frac{m_2\Theta_{k_1}(r)}{1+\Theta_{k_1}(r)})$, *then there exists a pair of quasisolution* $[\tilde{u}, \tilde{w}]$ *and* $[\hat{u}, \hat{w}]$ *of* (3.10) *with* $\tilde{u} \geqslant \hat{u}$ *and* $\tilde{w} \geqslant \hat{w}$. *Moreover,* $[\hat{u}, \tilde{u}] \times \{0\} \times [\hat{w}, \tilde{w}]$ *is a global attractor of* (1.1).

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